

## Solution of the two-dimensional wave equation by using wave polynomials

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**Abstract.** The paper demonstrates a specific power-series-expansion technique to solve approximately the two-dimensional wave equation. As solving functions (Trefftz functions) so-called wave polynomials are used. The presented method is useful for a finite body of certain shape geometry. Recurrent formulas for the wave polynomials and their derivatives are obtained in the Cartesian and polar coordinate system. The accuracy of the method is discussed and some examples are shown.

**Key words:** membrane vibrations, Trefftz method, wave equation, wave polynomials

### 1. Introduction

There are three groups of methods for solving partial differential equations. The first group contains the so-called analytical methods. It means that the solution exactly satisfies both the equation and all the given conditions. For example: integral-transformation method, separation-of-variables method, Green's function method and so on. Analytical solutions are not always useful for numerical calculations. For example, when the solution is given as Bessel-function series, there are often some problems with numerical convergence. Moreover, these methods are often useful only for simple body shapes. The second group contains methods such as the Finite Element Method or the Boundary Element Method. Then the solution approximately satisfies both the equation and all the given conditions and the shape does not have to be simple. In the third group the solution exactly satisfies the equation and approximately all given conditions. The Trefftz method belongs to this group. The method presented in this paper belongs to the third group. The key idea of the method is to find functions (polynomials) satisfying a given differential equation to be fitted to the governing initial and boundary conditions. In this sense it is a variant of the Trefftz method [1, 2].

The method originates from [3] but only for the case of one-dimensional heat-conduction problems. Heat polynomials were applied for solving unsteady heat conduction problems [4]. Both papers describe a one-dimensional problem for the heat equation in Cartesian coordinates. The method is continued in the Cartesian coordinate system in [5, 6], describing heat polynomials for two- and three-dimensional cases. Application of heat polynomials in polar and cylindrical coordinates can be found in [7–9]. A slightly different approach for one-dimensional heat polynomials is presented in [10].

In the scientific literature inverse problems are often presented. We can say that a problem is direct, when all the boundary and initial conditions known. In the inverse problems we are looking for an extrapolation of the function. The method which is described here is useful not only for direct problems but also for inverse ones. The application of this method to inverse heat-conduction problems is described in [5–9, 11, 12]. Reference [13] contains the highly interesting idea of using heat polynomials as a new type of finite-element base functions. All papers mentioned above refer to the heat-conduction

equation. Reference [14] deals with many other cases involving other differential equations such as the Laplace, Poisson and Helmholtz equations. Also the one-dimensional wave equation is solved there. Basically, the wave equation can be solved by different methods. Some of these are better suited for infinite bodies and others for finite bodies of simple shape. The method presented here is useful for finite bodies but the shape geometry is allowed to be relatively general.

Important for the application of the method are the properties of Taylor series:

$$f(x + \Delta x, y + \Delta y, t + \Delta t) = f(x, y, t) + \frac{df}{1!} + \frac{d^2 f}{2!} + \dots + \frac{d^N f}{N!} + R_{N+1} \quad (1)$$

where

$$d^n f = \left( \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y + \frac{\partial f}{\partial t} \Delta t \right)^n,$$

and  $R_{N+1}$  is the remainder term. Based on this, in Section 2 two-dimensional wave polynomials and their properties in the Cartesian coordinate system are considered. Section 3 contains two-dimensional wave polynomials in the polar coordinate system. In Section 4 the solution of the wave equation using wave polynomials is obtained. Section 5 contains some remarks on accuracy of approximation. In Section 6 some examples are discussed. Concluding remarks are given in Section 7.

## 2. Wave polynomials in a Cartesian coordinate system

There are two ways to obtain wave polynomials. The first is to use a 'generating function'. The second is to develop the wave equation in Taylor series. Both methods are equivalent and lead to the same wave polynomials.

### 2.1. GENERATING FUNCTION

Let us consider the nondimensional wave equation

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}. \quad (2)$$

The function

$$g = e^{i(ax+by+ct)} \quad (3)$$

satisfying Equation (2) when  $c^2 = a^2 + b^2$  is called a generating function for wave polynomials. The power series expansion for (3) is

$$e^{i(ax+by+ct)} = \sum_{n=0}^{\infty} \sum_{k=0}^n \sum_{l=0}^{n-k} S_{(n-k-l)kl}(x, y, t) a^{n-k-l} b^k c^l, \quad (4)$$

where  $S_{(n-k-l)kl}(x, y, t)$  are polynomials of variables  $x, y, t$ .

Substituting  $c^2 = a^2 + b^2$  in (4), we obtain

$$e^{i(ax+by+ct)} = \sum_{n=0}^{\infty} \sum_{k=0}^n \sum_{\substack{l=0 \\ l < 2}}^{n-k} R_{(n-k-l)kl}(x, y, t) a^{n-k-l} b^k c^l. \quad (5)$$

The real and imaginary parts of polynomials  $R$  satisfy Equation (2) and are called wave polynomials:

$$\begin{aligned} P_{(n-k-l)kl}(x, y, t) &= \Re(R_{(n-k-l)kl}(x, y, t)), \\ Q_{(n-k-l)kl}(x, y, t) &= \Im(R_{(n-k-l)kl}(x, y, t)), \end{aligned} \quad (6)$$

e.g.

$$\begin{aligned} P_{000}(x, y, t) &= 1, Q_{000}(x, y, t) = 0, & P_{100}(x, y, t) &= 0, Q_{100}(x, y, t) = x, \\ P_{010}(x, y, t) &= 0, Q_{010}(x, y, t) = y, & P_{001}(x, y, t) &= 0, Q_{001}(x, y, t) = t, \\ P_{200}(x, y, t) &= -\frac{x^2}{2} - \frac{t^2}{2}, Q_{200}(x, y, t) = 0, \\ P_{110}(x, y, t) &= -xy, Q_{110}(x, y, t) = 0, & P_{101}(x, y, t) &= -xt, Q_{101}(x, y, t) = 0, \\ P_{011}(x, y, t) &= -yt, Q_{011}(x, y, t) = 0, & P_{020}(x, y, t) &= -\frac{y^2}{2} - \frac{t^2}{2}, Q_{020}(x, y, t) = 0, \dots \end{aligned} \quad (7)$$

Note that there is no  $R_{002}$ , because  $l < 2$  (see formula (5)).

## 2.2. PARTIAL DERIVATIVES OF WAVE POLYNOMIALS

For numerical calculations we need the recurrent formulas of partial derivatives for wave polynomials (See section 6). To obtain these formulas we differentiate (5):

$$\frac{\partial g}{\partial x} = iag = \sum_{n=0}^{\infty} \sum_{k=0}^n \sum_{\substack{l=0 \\ l < 2}}^{n-k} \frac{\partial R_{(n-k-l)kl}}{\partial x} a^{n-k-l} b^k c^l.$$

Hence

$$\sum_{n=0}^{\infty} \sum_{k=0}^n \sum_{\substack{l=0 \\ l < 2}}^{n-k} iR_{(n-k-l)kl} a^{n-k-l+1} b^k c^l = \sum_{n=0}^{\infty} \sum_{k=0}^n \sum_{\substack{l=0 \\ l < 2}}^{n-k} \frac{\partial R_{(n-k-l)kl}}{\partial x} a^{n-k-l} b^k c^l$$

and

$$\frac{\partial R_{(n-k-l)kl}}{\partial x} = iR_{(n-k-l-1)kl},$$

so that finally

$$\frac{\partial P_{(n-k-l)kl}}{\partial x} = -Q_{(n-k-l-1)kl}, \quad \frac{\partial Q_{(n-k-l)kl}}{\partial x} = P_{(n-k-l-1)kl}. \quad (8)$$

Similarly we have

$$\frac{\partial P_{(n-k-l)kl}}{\partial y} = -Q_{(n-k-l)(k-1)l}, \quad \frac{\partial Q_{(n-k-l)kl}}{\partial y} = P_{(n-k-l)(k-1)l} \quad (9)$$

and

$$\begin{aligned} \frac{\partial P_{(n-k)k0}}{\partial t} &= -Q_{(n-k-2)k1} - Q_{(n-k)(k-2)1}, & \frac{\partial P_{(n-k-1)k1}}{\partial t} &= -Q_{(n-k-1)k0}, \\ \frac{\partial Q_{(n-k)k0}}{\partial t} &= P_{(n-k-2)k1} + P_{(n-k)(k-2)1}, & \frac{\partial Q_{(n-k-1)k1}}{\partial t} &= P_{(n-k-1)k0}. \end{aligned} \quad (10)$$

Starting values for the derivatives (8), (9) and (10) are obtained either from (7) or directly by putting zero instead of the polynomial in question in the right-hand sides of (8), (9) and (10) if any of its subscripts takes a negative value.

### 2.3. RECURRENT FORMULAS FOR WAVE POLYNOMIALS

Recurrent formulas are useful in numerical calculations. Theorem 1 enables one to get the wave polynomials  $P_{(n-k-l)kl}$  and  $Q_{(n-k-l)kl}$ .

**THEOREM 1.** *Let  $P_{000} = 1$  and  $Q_{000} = 0$ . Then, the polynomials*

$$P_{(n-k)k0} = \frac{1}{n}(-xQ_{(n-k-1)k0} - yQ_{(n-k)(k-1)0} - tQ_{(n-k-2)k1} - tQ_{(n-k)(k-2)1}), \quad (11)$$

$$P_{(n-k-1)k1} = \frac{1}{n}(-xQ_{(n-k-2)k1} - yQ_{(n-k-1)(k-1)1} - tQ_{(n-k-1)k0}), \quad (12)$$

$$Q_{(n-k)k0} = \frac{1}{n}(xP_{(n-k-1)k0} + yP_{(n-k)(k-1)0} + tP_{(n-k-2)k1} + tP_{(n-k)(k-2)1}), \quad (13)$$

$$Q_{(n-k-1)k1} = \frac{1}{n}(xP_{(n-k-2)k1} + yP_{(n-k-1)(k-1)1} + tP_{(n-k-1)k0}), \quad (14)$$

satisfy the wave Equation (2).

*Proof.* For relation (11) we assume that all polynomials on the right-hand side satisfy Equation (2). Substituting (11) in (2) we get

$$\begin{aligned} & x \left( \underbrace{\frac{\partial^2 Q_{(n-k-1)k0}}{\partial t^2} - \frac{\partial^2 Q_{(n-k-1)k0}}{\partial x^2} - \frac{\partial^2 Q_{(n-k-1)k0}}{\partial y^2}}_{=0} \right) \\ & + y \left( \underbrace{\frac{\partial^2 Q_{(n-k)(k-1)0}}{\partial t^2} - \frac{\partial^2 Q_{(n-k)(k-1)0}}{\partial x^2} - \frac{\partial^2 Q_{(n-k)(k-1)0}}{\partial y^2}}_{=0} \right) \\ & + t \left( \underbrace{\frac{\partial^2 Q_{(n-k-2)k1}}{\partial t^2} - \frac{\partial^2 Q_{(n-k-2)k1}}{\partial x^2} - \frac{\partial^2 Q_{(n-k-2)k1}}{\partial y^2}}_{=0} \right) \\ & + t \left( \underbrace{\frac{\partial^2 Q_{(n-k)(k-2)1}}{\partial t^2} - \frac{\partial^2 Q_{(n-k)(k-2)1}}{\partial x^2} - \frac{\partial^2 Q_{(n-k)(k-2)1}}{\partial y^2}}_{=0} \right) \\ & + 2 \frac{\partial Q_{(n-k-2)k1}}{\partial t} + 2 \frac{\partial Q_{(n-k)(k-2)1}}{\partial t} = 2 \frac{\partial Q_{(n-k-1)k0}}{\partial x} + 2 \frac{\partial Q_{(n-k)(k-1)0}}{\partial y}, \end{aligned}$$

whence

$$\frac{\partial Q_{(n-k-2)k1}}{\partial t} + \frac{\partial Q_{(n-k)(k-2)1}}{\partial t} = \frac{\partial Q_{(n-k-1)k0}}{\partial x} + \frac{\partial Q_{(n-k)(k-1)0}}{\partial y}$$

and according to (8), (9) and (10), we have

$$P_{(n-k-2)k0} + P_{(n-k)(k-2)0} = P_{(n-k-2)k0} + P_{(n-k)(k-2)0}.$$

This proves the theorem. The proof for (12), (13), and (14) is similar.

Similarly as before, starting values of the polynomials (11)–(14) can be obtained either from (7) or directly by putting zero instead of the polynomial in question in the right-hand side of (11)–(14) if any of its subscripts takes a negative value.

#### 2.4. EXPANSION OF THE WAVE EQUATION INTO TAYLOR SERIES

Similarly as for other equations [14], the wave polynomials can be obtained using Taylor series (1) for the function  $w$ . Let function  $w(x, y, t)$  satisfy the wave equation (2). We assume that  $w \in C^{N+1}$  in the neighborhood of  $(x_0, y_0, t_0)$ . Let  $\hat{x} = x - x_0$ ,  $\hat{y} = y - y_0$ ,  $\hat{t} = t - t_0$ . Then, the Taylor series for function  $w$  and for  $N = 2$  is

$$\begin{aligned} w(x, y, t) = & w(x_0, y_0, t_0) + \frac{\partial w}{\partial x} \hat{x} + \frac{\partial w}{\partial y} \hat{y} + \frac{\partial w}{\partial t} \hat{t} + \frac{\partial^2 w}{\partial x^2} \frac{\hat{x}^2}{2} + \frac{\partial^2 w}{\partial y^2} \frac{\hat{y}^2}{2} + \frac{\partial^2 w}{\partial t^2} \frac{\hat{t}^2}{2} \\ & + \frac{\partial^2 w}{\partial x \partial y} \hat{x} \hat{y} + \frac{\partial^2 w}{\partial x \partial t} \hat{x} \hat{t} + \frac{\partial^2 w}{\partial y \partial t} \hat{y} \hat{t} + R_3. \end{aligned} \quad (15)$$

Eliminating the derivative  $\partial^2 w / \partial t^2$  by Equation (2) yields

$$\begin{aligned} w(x, y, t) = & w(x_0, y_0, t_0) + \frac{\partial w}{\partial x} \hat{x} + \frac{\partial w}{\partial y} \hat{y} + \frac{\partial w}{\partial t} \hat{t} + \frac{\partial^2 w}{\partial x^2} \left( \frac{\hat{x}^2}{2} + \frac{\hat{t}^2}{2} \right) \\ & + \frac{\partial^2 w}{\partial y^2} \left( \frac{\hat{y}^2}{2} + \frac{\hat{t}^2}{2} \right) + \frac{\partial^2 w}{\partial x \partial y} \hat{x} \hat{y} + \frac{\partial^2 w}{\partial x \partial t} \hat{x} \hat{t} + \frac{\partial^2 w}{\partial y \partial t} \hat{y} \hat{t} + R_3. \end{aligned} \quad (16)$$

The coefficients succeeding the derivation terms on the right-hand side represent the non-zero wave polynomials (7). Similarly, we get polynomials for  $N = 3, 4, \dots$

### 3. Wave polynomials in polar coordinates

To obtain wave polynomials in polar coordinates,

$$x = r \cos \phi, \quad y = r \sin \phi, \quad (17)$$

we substitute (17) in (2) to get

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \phi^2}. \quad (18)$$

Then, to find the wave polynomials in polar coordinates, it is sufficient to substitute (17) in polynomials expressed in the Cartesian coordinate system. The polynomials obtained in that way satisfy Equation (18), e.g., for the non-zero polynomials (7)

$$\begin{aligned} P_{000}(r, \phi, t) &= 1, & Q_{100}(r, \phi, t) &= r \cos \phi, \\ Q_{010}(r, \phi, t) &= r \sin \phi, & Q_{001}(r, \phi, t) &= t, \\ P_{200}(r, \phi, t) &= -\frac{r^2 \cos^2 \phi}{2} - \frac{t^2}{2}, & P_{110}(r, \phi, t) &= -r^2 \cos \phi \sin \phi, \dots \end{aligned} \quad (19)$$

We can use the recurrent formulas (8)–(10) and (11)–(14) also in polar coordinates, keeping in mind that  $x = r \cos \phi$  and  $y = r \sin \phi$ .

#### 4. Wave-polynomial method

The wave-polynomial method discussed below belongs to the class of Trefftz methods. As a solution of wave Equation (2) we take a linear combination of wave polynomials which also satisfies Equation (2).

Let us denote the non-zero polynomials as

$$\begin{aligned} V_1 = 1, \quad V_2 = x, \quad V_3 = y, \quad V_4 = t, \quad V_5 = -\frac{x^2}{2} - \frac{t^2}{2}, \quad V_6 = -xy, \\ V_7 = -xt, \quad V_8 = -yt, \quad V_9 = -\frac{y^2}{2} - \frac{t^2}{2}, \quad \dots \end{aligned} \quad (20)$$

Obviously, we have one polynomial of order zero, three polynomials of order one, five polynomials of order two and so on.

An approximation for the solution of Equation (2) is

$$w \approx u = \sum_{n=1}^N c_n V_n. \quad (21)$$

Because all polynomials  $V_n$  satisfy Equation (2), also the linear combination  $u$  satisfies Equation (2). The coefficients  $c_n$  in (21) are chosen such that the error for fulfilling given boundary and initial conditions corresponding to Equation (2) is minimized (see examples).

#### 5. Accuracy of approximation

The wave-polynomial method is an approximation method. It is very important to know how big is the error of approximation. Moreover, this method should be convergent. It is easy to specify the error when in formula (21) all polynomials from order zero to  $K$  are taken, *e.g.*, for  $K = 0$ ,  $N = 1$ , for  $K = 1$ ,  $N = 1 + 3 = 4$ , for  $K = 2$ ,  $N = 1 + 3 + 5 = 9$  and so on. Then the error of approximation is equal to the remainder term in the Taylor series for function  $w$  (see relations (15) and (16)). This means that the wave-polynomial method is convergent if

$$\lim_{N \rightarrow \infty} R_N = 0$$

in the Taylor series of function  $w$ .

#### 6. Examples

We will show how the wave-polynomial method works by discussing two easy examples. The first concerns the vibrations of a square membrane. The second discusses the vibrations of a circular membrane.

##### 6.1. VIBRATIONS OF A SQUARE MEMBRANE

Let us consider the free vibrations of square membranes described by the following mathematical model:

– equation of motion:

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \quad (x, y) \in (0, 1) \times (0, 1), \quad t > 0, \quad (22)$$

– initial conditions:  
 $w(x, y, 0) = w_0(x, y) = x(x - 1)y(y - 1), \quad \frac{\partial w(x, y, 0)}{\partial t} = 0,$  (23)

– boundary conditions:  
 $w(0, y, t) = w(1, y, t) = w(x, 0, t) = w(x, 1, t) = 0.$  (24)

The exact solution of this problem is; (see [15, pp. 293–296])

$$w(x, y, t) = \frac{16}{\pi^2} \sum_{k,l=1}^{\infty} \frac{[(-1)^k - 1][(-1)^l - 1] \sin(k\pi x) \sin(l\pi y) \cos(\sqrt{k^2 + l^2}\pi t)}{k^3 l^3}. \quad (25)$$

All results below are obtained for  $k, l = 1, \dots, M$  where  $M = 40$ . The difference between the solutions for  $M = 41$  and  $M = 40$  is less than  $5 \times 10^{-6}$  in terms of the maximum norm.

The solution  $w(x, y, t)$  is approximated according to Equation (21):

$$w \approx u = \sum_{n=1}^N c_n V_n.$$

Because the function  $u$  satisfy the wave equation (linear combination of wave polynomials), we minimize  $\| w - u \|$  for the boundary and initial conditions. We look for an approximate solution  $u$  in the time interval  $(0, \Delta t)$ . Therefore, the coefficients  $c_n$  have to be chosen appropriately to minimize the functional

$$\begin{aligned} I = & \underbrace{\int_0^1 \int_0^1 [u(x, y, 0) - w_0(x, y)]^2 dy dx}_{\text{cond.(23)}} + \underbrace{\int_0^1 \int_0^1 \left[ \frac{\partial u(x, y, 0)}{\partial t} - \frac{\partial w(x, y, 0)}{\partial t} \right]^2 dy dx}_{\text{cond.(23)}} \\ & + \underbrace{\int_0^{\Delta t} \int_0^1 [u(0, y, t) - w(0, y, t)]^2 dy dt}_{\text{cond.(24)}} + \underbrace{\int_0^{\Delta t} \int_0^1 [u(1, y, t) - w(1, y, t)]^2 dy dt}_{\text{cond.(24)}} \\ & + \underbrace{\int_0^{\Delta t} \int_0^1 [u(x, 0, t) - w(x, 0, t)]^2 dx dt}_{\text{cond.(24)}} + \underbrace{\int_0^{\Delta t} \int_0^1 [u(x, 1, t) - w(x, 1, t)]^2 dx dt}_{\text{cond.(24)}}. \end{aligned} \quad (26)$$

A necessary condition to minimize the functional  $I$  is

$$\frac{\partial I}{\partial c_1} = \dots = \frac{\partial I}{\partial c_N} = 0. \quad (27)$$

The linear system of Equations (27) can be written as

$$AC = B \quad (28)$$

where  $C = [c_1, \dots, c_N]^T$  and the elements of matrices  $A$  and  $B$  are

$$\begin{aligned} a_{ij} = & \int_0^1 \int_0^1 V_i(x, y, 0) V_j(x, y, 0) dy dx + \int_0^1 \int_0^1 \frac{\partial V_i(x, y, 0)}{\partial t} \frac{\partial V_j(x, y, 0)}{\partial t} dy dx \\ & + \int_0^{\Delta t} \int_0^1 V_i(0, y, t) V_j(0, y, t) dy dt + \int_0^{\Delta t} \int_0^1 V_i(1, y, t) V_j(1, y, t) dy dt \end{aligned}$$

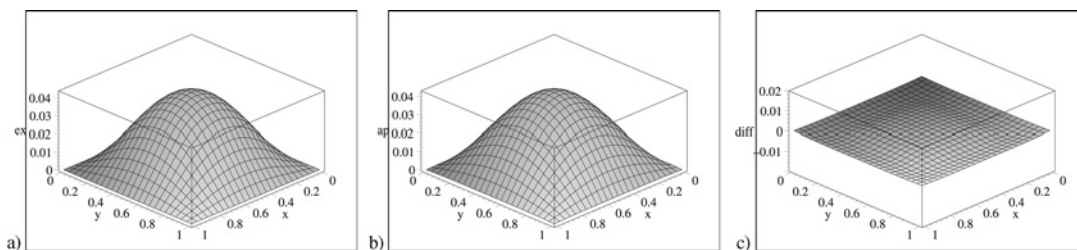


Figure 1. Solution for time  $t = 0.2$ : a) exact, b) approximation, c) difference.

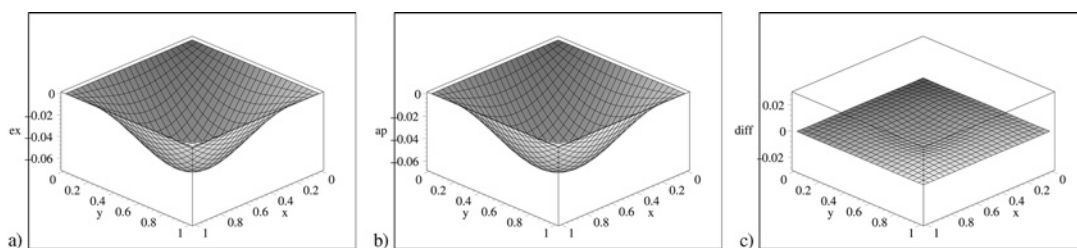


Figure 2. Solution for time  $t = 0.7$ : a) exact, b) approximation, c) difference.

$$\begin{aligned}
 & + \int_0^{\Delta t} \int_0^1 V_i(x, 0, t) V_j(x, 0, t) dx dt + \int_0^{\Delta t} \int_0^1 V_i(x, 1, t) V_j(x, 1, t) dx dt, \\
 b_i & = \int_0^1 \int_0^1 V_i(x, y, 0) w_0(x, y) dy dx.
 \end{aligned}$$

Note that  $a_{ij} = a_{ji}$  (matrix  $A$  is symmetric). This fact facilitates the calculations. From Equation (28) we obtain the coefficients  $c_n$ :

$$C = A^{-1}B.$$

In the time intervals  $(\Delta t, 2\Delta t)$ ,  $(2\Delta t, 3\Delta t), \dots$ , we proceed analogously. Here, the initial condition for time interval  $((m-1)\Delta t, m\Delta t)$  is the value of function  $u$  at the end of interval  $((m-2)\Delta t, (m-1)\Delta t)$ . All results below have been obtained for  $\Delta t = 1$ .

We obtain a good approximation in whole time interval  $(0, \Delta t)$ . For example, Figure 1 shows for time  $t = 0.2$  a) the exact solution, b) an approximation by polynomials from order 0 to 9, c) the difference between a) and b).

Figure 2 shows for time  $t = 0.7$ : a) the exact solution, b) an approximation by polynomials from order 0 to 9, c) the difference between a) and b). It is obvious that the approximation presented in the Figures 1 and 2 is very accurate. Figure 3 shows the exact result for the vibration as a function of time for the location  $x = 0.5$ ,  $y = 0.5$  and the approximation by polynomials from order 0 to a) 4, b) 6, c) 9.

In this case we can calculate the error of approximation in the norm  $H^1(\Omega)$ , where  $\Omega = (0, \Delta t)$  for point  $x = y = 0.5$ . The error is defined as:

$$\text{Error} = \sqrt{\frac{\int_0^{\Delta t} [w(0.5; 0.5; t) - u(0.5; 0.5; t)]^2 dt}{\int_0^{\Delta t} [w(0.5; 0.5; t)]^2 dt}}.$$

In approximation  $u$  we take all wave polynomials from order 0 to  $K$ . Table 1 shows the error which depends on the order  $K$ .



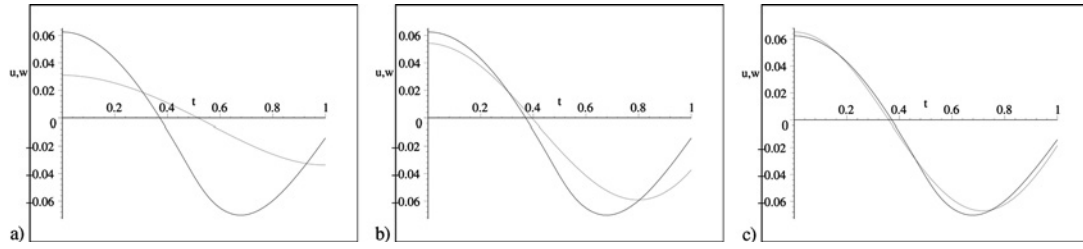


Figure 3. Exact solution versus time for point  $x = 0.5$ ,  $y = 0.5$  and approximation by polynomials from order 0 to a) 4, b) 6, c) 9.

Table 1. Error dependence of the polynomial order.

Order $K$	Error
4	0.6836
6	0.2845
9	0.0772

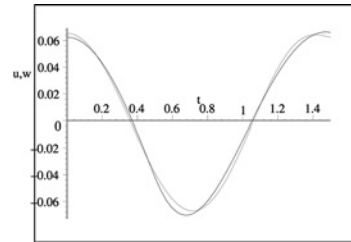


Figure 4. Extrapolation in time.

The error decreases when the number of the polynomials in approximation  $u$  increases. This means that the wave-polynomials method is convergent.

The wave-polynomials method is also useful for inverse problems. This means that we need an extrapolation of approximation  $u$  either in time or in space. The results are obtained for  $\Delta t = 1$ . Figure 4 shows the extrapolation in time for the vibration of point  $x = 0.5$ ,  $y = 0.5$  by polynomials up to order 9. Again it is obvious that the approximation and the extrapolation in time coincide very well.

## 6.2. VIBRATION OF A CIRCULAR MEMBRANE

Let us consider free vibrations of a circular membrane described (in Cartesian coordinates) by the following mathematical model:

– equation of motion:

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}, \quad x^2 + y^2 < 1, \quad t > 0, \quad (29)$$

– initial conditions:

$$w(x, y, 0) = w_0(x, y) = \frac{1}{20}(1 - x^2 - y^2), \quad \frac{\partial w(x, y, 0)}{\partial t} = 0, \quad (30)$$

– a boundary condition:

$$w(x, y, t) = 0, \quad x^2 + y^2 = 1. \quad (31)$$

Transforming the problem into polar coordinates, we get

– equation of motion:

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \phi^2}, \quad (r, \phi) \in (0, 1) \times (0, 2\pi), \quad t > 0, \quad (32)$$

– initial conditions:

$$w(r, \phi, 0) = w_0(r, \phi) = \frac{1-r^2}{20}, \quad \frac{\partial w(r, \phi, 0)}{\partial t} = 0, \quad (33)$$

– a boundary condition:

$$w(1, \phi, t) = 0, \quad (34)$$

The exact solution for this problem, using Bessel functions  $J_0$  and  $J_1$ , is; (see [15, pp. 296–298])

$$w(r, t) = \sum_{n=1}^{\infty} \frac{J_0(k_n r) \cos(k_n t)}{10J_1^2(k_n)} \int_0^1 r(1-r^2)J_0(k_n r)dr,$$

where  $k_n$  are positive roots of the equation  $J_0(k) = 0$ . Unfortunately, for a large number of these functions in the polar coordinate system there are some problems with convergence.

Let us use the solution (21):

$$w \approx u = \sum_{n=1}^N c_n V_n,$$

where  $V_n$  are the wave polynomials in polar coordinates. We look for an approximate solution in the time interval  $(0, \Delta t)$ . The coefficients  $c_n$  have to be computed such that the functional

$$\begin{aligned} I = & \underbrace{\int_0^1 \int_0^{2\pi} [u(r, \phi, 0) - w_0(r, \phi)]^2 r d\phi dr}_{\text{cond.(33)}} + \underbrace{\int_0^1 \int_0^{2\pi} \left[ \frac{\partial u(r, \phi, 0)}{\partial t} - \frac{\partial w(r, \phi, 0)}{\partial t} \right]^2 r d\phi dr}_{\text{cond.(33)}} \\ & + \underbrace{\int_0^{\Delta t} \int_0^{2\pi} [u(1, \phi, t) - w(1, \phi, t)]^2 r d\phi dt}_{\text{cond.(34)}} \end{aligned} \quad (35)$$

is minimized. To obtain the coefficients  $c_n$ , the procedure is the same as in Section 6.1. Here the elements of the matrices  $A$  and  $B$  are:

$$\begin{aligned} a_{ij} = & \int_0^1 \int_0^{2\pi} V_i(r, \phi, 0) V_j(r, \phi, 0) r d\phi dr + \int_0^1 \int_0^{2\pi} \frac{\partial V_i(r, \phi, 0)}{\partial t} \frac{\partial V_j(r, \phi, 0)}{\partial t} r d\phi dr \\ & + \int_0^{\Delta t} \int_0^{2\pi} V_i(1, \phi, t) V_j(1, \phi, t) r d\phi dt \\ b_i = & \int_0^1 \int_0^{2\pi} V_i(r, \phi, 0) w_0(r, \phi) r d\phi dr. \end{aligned}$$

Similarly as in Section 6.1 matrix  $A$  is symmetric.

All results below are obtained for  $\Delta t = 1$ . In this case the solution is independent of the angle  $\phi$ . Figure 5 shows the vibrations as dependent on radius and time: a) the exact solution, b) an approximation by polynomials from order 0 to 9, c) the difference between a) and b).

It is obvious that the approximation result is very good.

Figure 6 shows the exact solution of the vibration versus time for the midpoint  $r = 0$  and an approximation by polynomials from order 0 to a) 3, b) 5, c) 9. We can calculate the error of approximation in the norm  $H^1(\Omega)$ , where  $\Omega = (0, \Delta t)$  for point  $r = 0$ . The error is defined as:

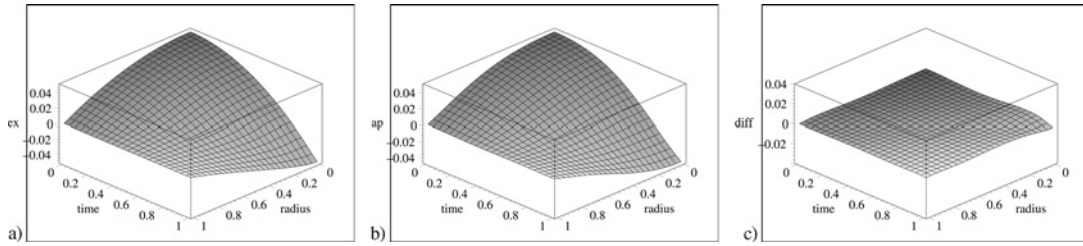


Figure 5. Solution versus time  $t \in [0, 1]$ : a) exact, b) approximation, c) difference.

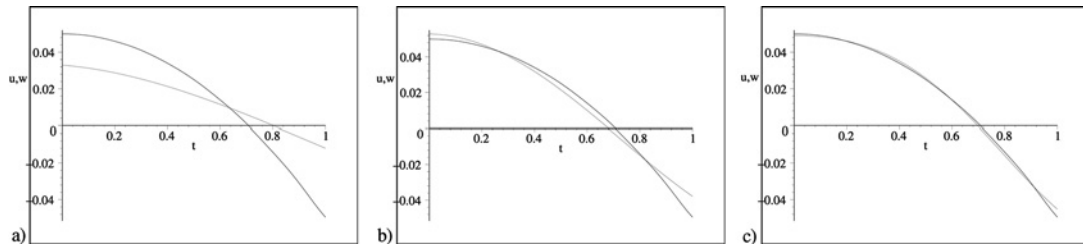


Figure 6. Vibration versus time for point  $r = 0$  by polynomials from order 0 to: a) 3, b) 5, c) 9.

Table 2. Error dependence of the polynomial order.

Order $K$	Error
4	0.4848
6	0.1059
9	0.0361

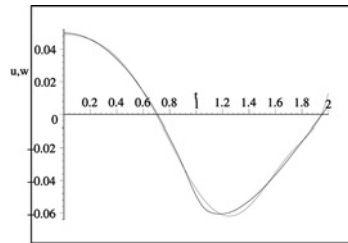


Figure 7. Extrapolation in time.

$$\text{Error} = \sqrt{\frac{\int_0^{\Delta t} [w(r, \phi, t) - u(r, \phi, t)]^2 dt}{\int_0^{\Delta t} [w(r, \phi, t)]^2 dt}}$$

Here the angle  $\phi$  is arbitrary (e.g  $\phi = 0$ ), because the solution is independent of the angle. In the approximation  $u$  we take all wave polynomials from order 0 to  $K$ . Table 1 shows the error dependence of order  $K$ . Obviously, the error decreases when the number of the polynomials in approximation  $u$  increases. It means that the wave-polynomials method is convergent in polar coordinate system.

The extrapolation of approximation  $u$  can be used also in polar coordinates. Figure 7 shows the extrapolation in time for the vibration of point  $r = 0$  by polynomials up to order 9 (the results are obtained for  $\Delta t = 1$ ).

Again, it is obvious that approximation and extrapolation in time coincide very well.

### 7. Concluding remarks

A new technique for solving the two-dimensional wave equation has been developed. Our main result is the derivation of the formulas for the wave polynomials (satisfying a wave equation) and their derivatives. The wave-polynomial method presented in this paper is a straightforward method for solving wave equations in finite bodies. This method is also useful when the shape of the body is complicated.

The only condition is that the shape geometry is given or can be approximated in polynomial form. Then, the evaluation basically remains unchanged and the only difficulty for such a shape may be the calculation of the integrals determining the coefficients  $c_n$  – for most shapes this does not create any problems. The method is convergent and the error is equal to the remainder term in the Taylor series. The simple examples presented show that the obtained approximations of the exact solutions are very good, both in Cartesian and polar coordinates. Especially, in the polar coordinate system this method avoids ill-convergent Bessel functions. The solution, being a linear combination of wave polynomials, exactly satisfies the wave equation and approximately initial and boundary conditions. It is important, that this method can be used for extrapolation. Therefore, the wave-polynomial method can also be applied to inverse problems. Wave polynomials can be used as finite-element base functions. This will be the subject of another paper.

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